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Sedimentation and suspension flows: Historical perspective and some recent developments

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Abstract. Sedimentation and suspension flows play an important role in modern technology. This special issue joins nine recent contributions to the mathematics of these processes. The Guest Editors provide a concise account of the contributions to research in sedimentation and thickening that were made during the 20th century with a focus on the different steps of progress that were made in understanding batch sedimentation and continuous thickening processes in mineral processing. A major breakthrough was Kynch's kinematic sedimentation theory published in 1952. Mathematically, this theory gives rise to a nonlinear first-order scalar conservation law for the local solids concentration. Extensions of this theory to continuous sedimentation, flocculent and polydisperse suspensions, vessels with varying cross-section, centrifuges and several space dimensions, as well as its current applications are reviewed.

Key words: conservation laws, emulsions, mathematical models, multiphase flow, sedimentation, suspensions, thickening

1. Introduction

Sedimentation and suspension flows involve the mechanics, flow and transport properties of mixtures of fluids and solids, droplets or bubbles. Fundamental aspects of sedimentation and suspension flows include properties of suspensions and emulsions (rheology, particle size and shape, particle-particle interaction, surface characteristics, yield stress, concentration, viscosity), individual particles (orientation and surfactants), and sediments and porous cakes (permeability, porosity and compressibility). They are of critical importance, especially in the field of solid-liquid separations in the chemical, mining, pulp and paper, wastewater, food, pharmaceutical, ceramic and other industries. Mathematical models for these processes are of obvious theoretical and practical importance. It is the purpose of this special issue to present nine recent contributions that develop different aspects of the mathematics involved in modelling sedimentation and suspension flows.

An outstanding reason for publishing this special issue just now is the fiftieth anniversary of the celebrated paper *A Theory of Sedimentation* by G.J. Kynch [1], submitted in 1951 and published in 1952. Before providing an introduction to the papers of this special issue, we therefore use the opportunity to present a brief account of the historical perspective that led to this theory, and its diverse reverberations in mathematics and the applied sciences.

Some of the contributions were presented at a workshop *Mathematical Problems in Suspension Flow*, which took place at the University of Stuttgart, October 9–11, 1999. We would like to thank the European Science Foundation (ESF) for generous support within the programme *Applied Mathematics in Industrial Flow Problems* (AMIF), which made this event

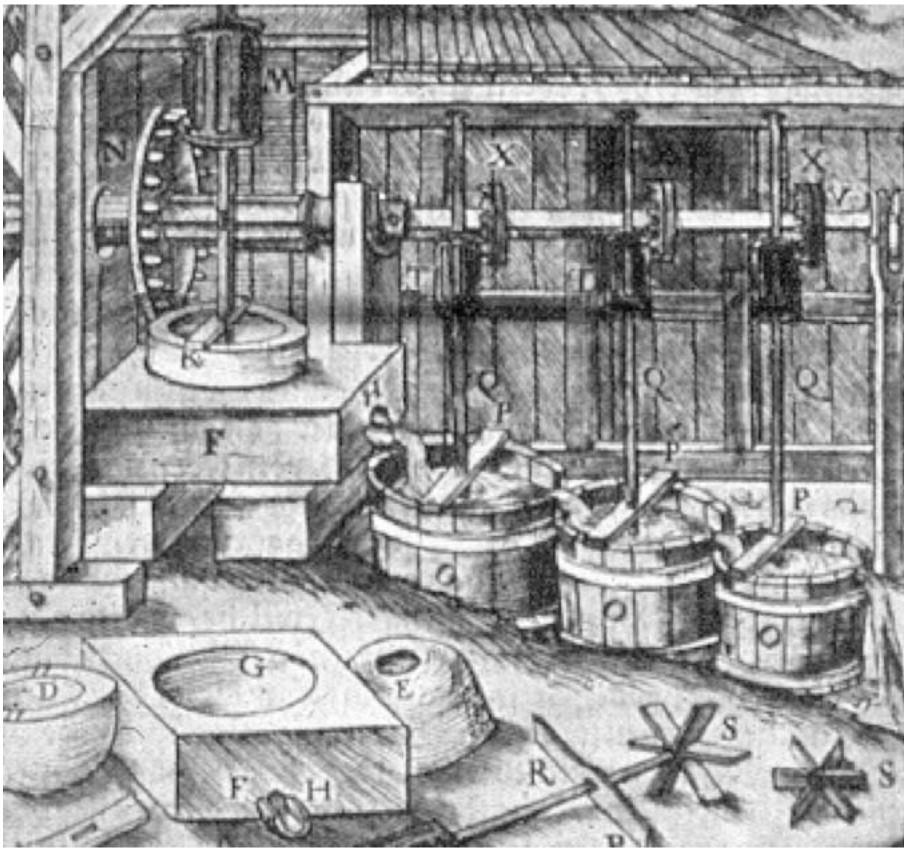

Figure 1. Washing and settling according to Agricola (1556) [4].

possible. We would also like to thank Professor Dr H. K. Kuiken for giving us the possibility to publish this issue as Guest Editors of the *Journal of Engineering Mathematics.*

2. On the history of sedimentation research

The controlled sedimentation of suspensions of small particles in a fluid, also referred to as *thickening* or *clarification* depending on whether either the concentrated sediment or the clarified liquid is considered as the main result of the process, is not a modern undertaking, and was already utilized by the ancient Egyptians, who dug for and washed gold. The earliest written reference for crushing and washing ores in Egypt is that of Agatharchides, a Greek geographer who lived 200 years before Christ. Ardaillon [2] described in 1897 the process used in the extensive installations for crushing and washing ores in Greece between the fifth and the third centuries BC. Wilson [3] describes mining of gold and copper in the Mediterranean from the fall of the Egyptian dynasties right to the Middle Ages and the Renaissance. It is evident that, by using washing and sifting processes, the ancient Egyptians and Greeks and the medieval Germans and Cornishmen knew the practical effect of the difference in specific gravity of the various components of an ore and used sedimentation in operations that can now be identified as classification, clarification and thickening. There is also evidence that in the early days no clear distinction was made between these three operations.

Agricola's book *De Re Metallica* [4] formed the first major contribution to the development and understanding of the mining industry. It was published in Latin in 1556, and shortly after translated into German and Italian. It describes several methods of washing metallic ores. In particular, Agricola describes settling tanks used as classifiers, jigs and thickeners and settling ponds used as thickeners or clarifiers (see Figure 1), that were operated in a batch or semi-continuous manner. Agricola's textbook continued to be the leading textbook for miners and metallurgists for at least three hundred years. Although it was, of course, far from providing anything like a theory of sedimentation, it formed an important step in the development of mineral processing from unskilled labour to craftsmanship and eventually an industry governed by scientific discipline [5].

The processes of classification, clarification and thickening all involve the sedimentation of small particles in a fluid. However, while clarification deals with very dilute suspensions, classification and thickening are forced to use more concentrated pulps. That the flow of a dilute suspension can be approximated by that of a clear liquid is probably the reason why clarification was the first of these operations amenable to mathematical description. The work by Hazen in 1904 [6] was the first analysis of factors affecting the settling of solid particles from dilute suspensions in water. It shows that detention time is not a factor in the design of settling tanks, but rather that the portion of solid removed was proportional to the surface area of the tank and to the settling properties of the solid matter, and inversely proportional to the flow through the tank.

The invention of the continuous thickener by John V. N. Dorr in 1905 [7] can be mentioned as the starting point of the modern thickening era and rigorous scientific research. The invention of the Dorr thickener made the continuous dewatering of a dilute pulp possible, whereby a regular discharge of a thick pulp of uniform density took place concurrently with overflow of clarified solution. Scraper blades or rakes, driven by a suitable mechanism, rotating slowly over the bottom of the tank, which usually slopes gently toward the center, move the material as fast as it settles without enough agitation to interfere with the settling.

The introduction of the continuous thickener initiated scientific research on sedimentation and thickening in the modern sense of establishing a quantitative theory that would be able to explain the thickening process and provide a design procedure for sedimentation tanks. For example, in 1912 R.T. Mishler was the first to show by experiments that the rate of settling of slimes is different for dilute than for concentrated suspensions [8]. While the settling speed of dilute slimes is usually independent of the depth of the settling column, a different law governs extremely thick slimes, and sedimentation increases with the depth of the settling column. In 1918 he devised formulas by means of which laboratory results could be used in continuous thickeners [9]. These formulas represent macroscopic balances of water and solids in the thickener and are explicitly stated in [5, 10].

The early researchers at the beginning of the last century soon recognized that it was not sufficient to study the global operating variables, and that it was more important to investigate the mechanisms effective in the interior of the vessels, most notably the settling velocities of particles, the formation of sediments and the evolution of concentration fronts. Experimental efforts in this direction were apparently first made by Clark in 1915 [11]. He carefully measured concentrations in a thickener with conical bottom, a configuration that clearly gives rise to at least a two-dimensional flow.

Clark's measurements partly stimulated the well-known and to this day frequently cited paper by Coe and Clevenger, which appeared in 1916 [12]. Coe and Clevenger were the first to recognize that the settling process of a flocculent suspension gives rise to four different and

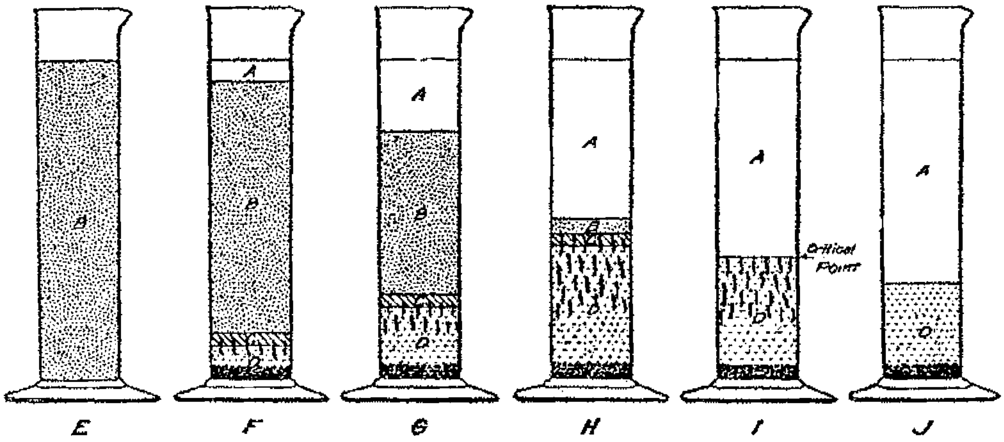

Figure 2. Settling of a flocculent suspension as illustrated by Coe and Clevenger (1916) [12], showing the clear water zone (A), the zone in which the suspension is at its initial concentration (B), the transition zone (C) and the compression zone (D).

well-distinguishable zones. From top to bottom, they determined a *clear water zone*, a zone in which the suspension is present at its *initial concentration*, a *transition zone* and a *compression zone*; see Figure 2. Coe and Clevenger reported settling experiments with a variety of materials showing this behaviour. Furthermore, they were also the first to use the observed batch settling data in a laboratory column for the design of an industrial thickener, and in particular devised a formula for the required cross-sectional area of a continuous thickener at given solids-handling capacity.

In the next two decades, several authors [13–16] made efforts to model the settling of suspensions by extending the Stokes formula, which states that the final settling velocity of a sphere of diameter d and density ϱ_s in an unbounded fluid of density ϱ_f and dynamic viscosity μ_f is given by

$$u_\infty = -\frac{(\varrho_s - \varrho_f) g d^2}{18\mu_f}, \tag{1}$$

where g is the acceleration of gravity, but no further important contributions were made until the 1940s. In 1940 E.W. Comings published a paper [17] which was the first to show remarkably accurate measurements of solids concentration profiles in a continuous thickener, while all previous treatments had been concerned with observations of the suspension-supernate and sediment-suspension interfaces only (with the exception of Clark [11]). Results of the numerous theses he guided at the University of Illinois on continuous sedimentation were summarized in 1954 in an important paper by Comings *et al.* [18]. In particular, in a continuous thickener four zones are identified: the clarification zone at the top, the settling zone underneath, the upper compression zone further down and the rake-action zone at the bottom. It is worth mentioning that the paper [18] did not yet take into account Kynch's sedimentation theory published two years earlier [1], which will be discussed below.

Other contributions of practical importance are the series of papers by H. H. Steinour that appeared in 1944 [19–21], which are the first to relate observed macroscopic sedimentation rates to microscopic properties of solid particles, and the work of Roberts, published in 1949 [22]. Roberts advanced the empirical hypothesis that the rate at which water is eliminated from a pulp in compression is at all times proportional to the amount that is left, which can be eliminated up to infinite time:

$$\mathcal{D} - \mathcal{D}_\infty = (\mathcal{D}_0 - \mathcal{D}_\infty)\exp(-Kt), \tag{2}$$

where $\mathcal{D}_0$, $\mathcal{D}$ and $\mathcal{D}_\infty$ are the dilutions at times zero and t and at infinite time, respectively. The equation has been used until today for the determination of the critical concentration.

3. Kynch's theory of sedimentation

All the papers cited so far were solely based on a macroscopic balance of the solid and the fluid and on the observation of the different zones in the thickener. No underlying sedimentation 'theory' existed in the modern sense of a partial differential equation whose solution could at least approximately explain the observed sedimentation behaviour.

G.J. Kynch, a mathematician at the University of Birmingham in Great Britain, presented in 1951 his celebrated paper *A theory of sedimentation* [1]. He proposed a kinematical theory of sedimentation based on the propagation of kinematic waves in an idealized suspension. The suspension is considered as a continuum and the sedimentation process is represented by the continuity equation of the solid phase:

$$\frac{\partial\phi}{\partial t} + \frac{\partial f_{\mathrm{bk}}(\phi)}{\partial z} = 0, \quad 0 \le z \le L,\ t > 0, \tag{3}$$

where ϕ is the local volume fraction of solids as a function of height z and time t, and $f_{\mathrm{bk}}(\phi) = \phi v_{\mathrm{s}}$ is the *Kynch batch flux density function*, where v_{s} is the solids-phase velocity. The basic assumption is that the local solid-liquid relative velocity is a function of the solids volumetric concentration ϕ only, which for batch sedimentation in a closed column is equivalent to stating that $v_{\mathrm{s}} = v_{\mathrm{s}}(\phi)$. For the sedimentation of an initially homogeneous suspension of concentration ϕ_0, Equation (3) is considered together with the initial condition

$$\phi(z,0) = \begin{cases} 0 & \text{for } z = L, \\ \phi_0 & \text{for } 0 < z < L, \\ \phi_{\max} & \text{for } z = 0, \end{cases} \tag{4}$$

where it is assumed that the function f_{bk} satisfies $f_{\mathrm{bk}}(\phi) = 0$ for $\phi \le 0$ or $\phi \ge \phi_{\max}$ and $f_{\mathrm{bk}}(\phi) < 0$ for $0 < \phi < \phi_{\max}$, where $\phi_{\max}$ is the maximum solids concentration. Kynch [1] shows that knowledge of the function f_{bk} is sufficient to determine the sedimentation process, *i.e.* the solution $\phi = \phi(z,t)$, for a given initial concentration ϕ_0, and that the solution can be constructed by the method of characteristics.

To describe the batch-settling velocities of particles in real suspensions of small particles, numerous material specific constitutive equations for $v_{\mathrm{s}} = v_{\mathrm{s}}(\phi)$ or $f_{\mathrm{bk}}(\phi) = \phi v_{\mathrm{s}}(\phi)$ were proposed. These can all be regarded as extensions of the Stokes formula (1). The most frequently used is the two-parameter equation of Richardson and Zaki [23]:

$$f_{\mathrm{bk}}(\phi) = u_\infty \phi(1-\phi)^n, \quad n > 1. \tag{5}$$

This equation has the inconvenience that the settling velocity becomes zero at the solids concentration $\phi = 1$, while experimentally this occurs at a maximum concentration $\phi_{\max}$ between 0·6 and 0·7.

Michaels and Bolger [24] proposed the following three-parameter alternative:

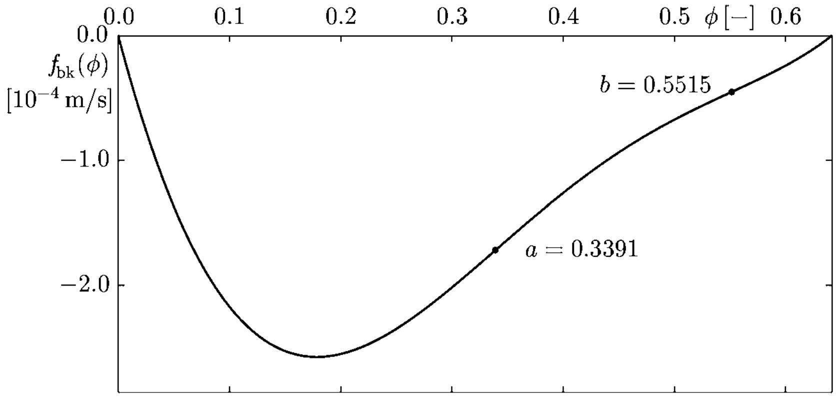

Figure 3. Flux density function for glass beads with two inflection points a and b.

$$f_{bk}(\phi) = u_\infty \phi (1 - \phi/\phi_{max})^n, \quad n > 1, \tag{6}$$

where the exponent $n = 4{\cdot}65$ turned out to be suitable for rigid spheres.

For equally sized glass spheres, Shannon *et al.* [25] determined the following equation by fitting a fourth-order polynomial to experimental measurements, see Figure 3:

$$f_{bk}(\phi) = \phi(-0{\cdot}3384 + 1{\cdot}3767\phi - 1{\cdot}6228\phi^2 - 0{\cdot}1126\phi^3 + 0{\cdot}90225\phi^4) \times 10^{-2}\text{m/s}.$$

Experiments aiming at verifying the validity of Kynch's theory have repeatedly been conducted up to the present day [26–28].

4. Extensions, mathematical analysis and applications

4.1. Mathematical analysis

To construct the solution of the initial-value problem (3), (4), the method of characteristics is employed. This method is based on the propagation of $\phi_0(z_0)$, the initial value prescribed at $z = z_0$, at constant speed $f'_{bk}(\phi_0(z_0))$ in a z vs. t diagram. These straight lines, the *characteristics*, might intersect, which makes solutions of Equation (3) discontinuous in general. This is due to the nonlinearity of the flux-density function f_{bk}. In fact, even for smooth initial data, a scalar conservation law with a nonlinear flux density function may produce discontinuous solutions, as the well-known example of Burgers' equation illustrates; see Le Veque [29].

To outline the main properties of discontinuous solutions of scalar equations like Equation (3), consider the *Riemann problem*, where an initial function

$$\phi_0(z) = \begin{cases} \phi_0^+ & \text{for } z > 0, \\ \phi_0^- & \text{for } z < 0 \end{cases} \tag{7}$$

consisting of just two constants is prescribed. Obviously, the initial-value problem (3), (4) consists of two adjacent Riemann problems producing two 'fans' of characteristics and discontinuities, which in this case start to interact after a finite time t_1.

At discontinuities, Equation (3) is not satisfied and is replaced by the *Rankine-Hugoniot condition*, which states that the local propagation velocity $\sigma(\phi^+, \phi^-)$ of a discontinuity between the solution values ϕ^+ above and ϕ^- below the discontinuity is given by

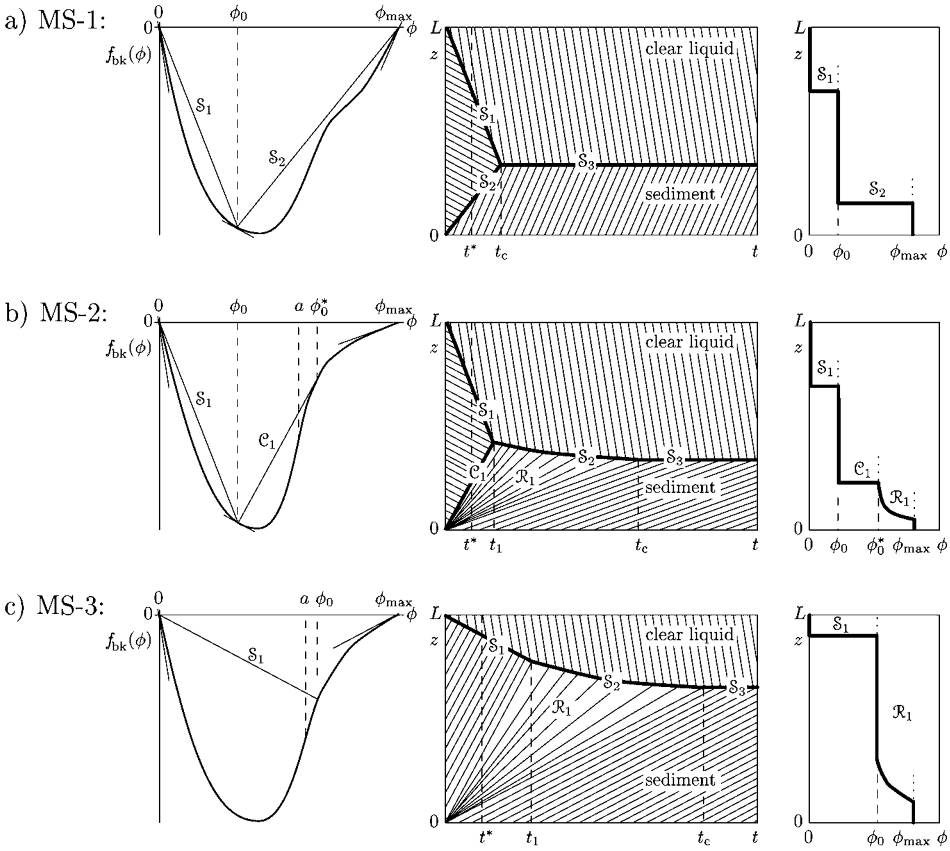

Figure 4. Modes of sedimentation MS-1 to MS-3. From the left to the right, the flux plot, the settling plot showing characteristics and shock lines, and one concentration profile (for $t = t^*$) are shown for each mode. Chords in the flux plots and shocks in the settling plots having the same slopes are marked by the same symbols. Slopes of tangents to the flux plots occurring as slopes of characteristics in the settling plots are also indicated.

$$\sigma(\phi^+, \phi^-) = \frac{f_{\text{bk}}(\phi^+) - f_{\text{bk}}(\phi^-)}{\phi^+ - \phi^-}. \tag{8}$$

However, discontinuous solutions satisfying (3) at points of continuity and the Rankine–Hugoniot condition (8) at discontinuities are, in general, not unique. For this reason, an additional selection criterion is necessary to select the physically relevant discontinuous solution. One of these entropy criteria, which determine the unique weak solution, is Oleĭnik's jump condition requiring that

$$\frac{f_{\text{bk}}(\phi) - f_{\text{bk}}(\phi^-)}{\phi - \phi^-} \geq \sigma(\phi^+, \phi^-) \geq \frac{f_{\text{bk}}(\phi) - f_{\text{bk}}(\phi^+)}{\phi - \phi^+} \quad \text{for all } \phi \text{ between } \phi^- \text{ and } \phi^+ \tag{9}$$

is valid. This condition has an instructive geometrical interpretation: it is satisfied if and only if, in an f_{bk} vs. ϕ plot, the chord joining the points $(\phi^+, f_{\text{bk}}(\phi^+))$ and $(\phi^-, f_{\text{bk}}(\phi^-))$ remains above the graph of f_{bk} for $\phi^+ < \phi^-$ and below the graph for $\phi^+ > \phi^-$.

Discontinuities satisfying both (8) and (9) are called *shocks*. If, in addition,

$$f'_{\text{bk}}(\phi^-) = \sigma(\phi^+, \phi^-) \quad \text{or} \quad f'_{\text{bk}}(\phi^+) = \sigma(\phi^+, \phi^-) \tag{10}$$

is satisfied, the shock is called a *contact discontinuity*. In that case, the chord is tangent to the graph of $f_{\rm bk}$ in at least one of its endpoints.

Consider Equation (3) together with the Riemann data (7). If we assume (for simplicity) that $\phi_0^- < \phi_0^+$ and that $f'_{\rm bk}(\phi) > 0$ for $\phi_0^- \le \phi \le \phi_0^+$, it is easy to see that no shock can be constructed between ϕ_0^- and ϕ_0^+. In that case, the Riemann problem has a continuous solution

$$\phi(z,t) = \begin{cases} \phi_0^+ & \text{for } z > f'_{\rm bk}(\phi_0^+)t, \\ (f'_{\rm bk})^{-1}(z/t) & \text{for } f'_{\rm bk}(\phi_0^-)t \le z \le f'_{\rm bk}(\phi_0^+)t, \\ \phi_0^- & \text{for } z < f'_{\rm bk}(\phi_0^-)t, \end{cases} \tag{11}$$

where $(f'_{\rm bk})^{-1}$ is the inverse of $f'_{\rm bk}$ restricted to the interval $[\phi_0^-, \phi_0^+]$. This solution is called a *rarefaction wave* and is the unique physically relevant weak solution of the Riemann problem.

A piecewise continuous function satisfying the conservation law (3) at points of continuity, the initial condition (4), and the Rankine–Hugoniot condition (8) and Oleĭnik's condition (9) at discontinuities is unique. For the problem of sedimentation of an initially homogeneous suspension, giving rise to two adjacent Riemann problems only, such a solution can be explicitly constructed by the method of characteristics. For example, for a flux-density function $f_{\rm bk}$ with exactly one inflection point, there are three qualitatively different solutions, denoted according to Kynch [1] as *Modes of Sedimentation*, shown in Figure 4. A particularly concise overview of the seven modes of sedimentation for flux density functions $f_{\rm bk}$ having at most two inflection points is given by Bürger and Tory [30].

It is interesting to note that Kynch [1] did not construct these complete solutions; rather, he presented a discussion of stable and instable kinematic discontinuities relying on physical insight, and postulated that only stable discontinuities should occur. Based on these considerations Wallis in 1962 [31] and Grassmann and Straumann in 1963 [32] constructed the complete discontinuous solutions as sketched in our Figure 4. At the same time the mathematical analysis of conservation laws like (3) was started. One of the results was condition (9). The formulation of admissibility conditions for more general discontinuous solutions (not necessarily piecewise differentiable ones) led to the concept of *entropy-weak solutions*. One of the most frequently cited works in this framework is Kružkov's paper [33], published in 1970, which presents a general existence and uniqueness result. Kružkov's approach is also well documented in any of the newly released textbooks on the analysis of conservation laws [34, 35, 36].

In 1984, M. C. Bustos in her thesis [37] appropriately embedded Kynch's theory into the state of the art of mathematical analysis. In a series of papers, summarized in Chapter 7 of [10], it was confirmed that the known solutions constructed in [31, 32] are indeed special cases of entropy-weak solutions. Utilizing the method of characteristics and applying the theory developed by Ballou [38], Cheng [39, 40] and Liu [41], it was possible to extend the construction of modes of sedimentation to the Kynch batch flux-density function with two or more inflection points.

4.2. EXTENSIONS

4.2.1. *Continuous sedimentation*

In 1975 Petty [42] made an attempt to extend Kynch's theory from batch to continuous sedimentation. If $q = q(t)$ is defined as the volume flow rate of the mixture per unit area of the sedimentation vessel, Kynch's equation for continuous sedimentation can be written as

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z}\big(q(t)\phi + f_{\mathrm{bk}}(\phi)\big) = 0. \tag{12}$$

Starting from Petty's model [42], Bustos, Concha and Wendland [43] studied a very simple model for continuous sedimentation, in which Equaion (12) is restricted to a space interval $[0, L]$, corresponding to a cylindrical vessel, and where the upper end $z = L$ is identified with a feed inlet and the lower $z = 0$ with a discharge outlet. The vessel is assumed to be fed continuously with feed suspension at the inlet (surface source) and to be discharged continuously through the outlet (surface sink). The overflow of clear liquid is not explicitly modelled. The volume average velocity $q = q(t)$ is a prescribed control function determined by the discharge opening. In [43] Equation (12) is provided with Dirichlet boundary conditions at $z = 0$ and $z = L$ and appropriately studied in the framework of entropy boundary conditions [44].

Unfortunately, for practical use, this model has some severe shortcomings. Among them is the lack of a global conservation principle due to the use of Dirichlet boundary conditions. It is preferable to replace the boundary conditions at the ends of the vessel by transitions between the transport flux $q\phi$ and the composite flux $q\phi + f_{\mathrm{bk}}(\phi)$, such that the problem is reduced to a pure initial-value problem. Moreover, in a realistic model the feed suspension should enter at a feed level located between the overflow outlet at the top and the discharge outlet at the bottom. This gives rise to a conservation law with a flux-density-function that is discontinuous at three different heights. Particularly thorough analyses of such ideal clarifier-thickener models were presented by Diehl in a series of papers (see [45] and the references cited by Diehl in his contribution to this issue). Recently Bürger *et al.* [46] showed that the front-tracking method [47] can be employed as an efficient simulation tool for continuous sedimentation processes in ideal clarifier-thickener units.

4.2.2. *Flocculent suspensions*

Experience by several authors, most notably by Scott [48], demonstrated that, while Kynch's theory accurately predicts the sedimentation behaviour of suspensions of equally sized small rigid spherical particles, this is not the case for flocculent suspensions forming compressible sediments. For such mixtures a kinematic model is no longer sufficient and one needs to take into account dynamic effects, in particular the concept of effective solid stress. Starting from the local mass and linear momentum balances for the solid and the fluid, introducing constitutive assumptions and simplifying the resulting equations due to a dimensional analysis, one then obtains a strongly degenerate convection-diffusion equation, *i.e.* Equation (3) with an additional degenerating second-order diffusion term, as a suitable extension of Kynch's theory [49]. Such an equation is studied in Bürger and Karlsen's contribution to this issue.

4.2.3. *Polydisperse suspensions*

Kinematic models of sedimentation can also be formulated for suspensions with small spherical particles belonging to a finite number N of species that differ in size or density. Specifying for each species the solid-fluid relative or slip velocity, or equivalently a scalar flux-density function, leads to a nonlinear coupled system of N scalar first-order conservation laws for the N concentration values of the solid species. The difficulty is that it is by no means obvious how to generalize, for example, the scalar Richardson and Zaki flux-density function, Equation (5), to a polydisperse system. Two mathematical models for polydisperse sedimentation that can be expressed as such first-order systems of conservation laws are considered in this issue in the paper by Bürger, Fjelde, Höfler and Karlsen.

In a recent paper [50], we show that, depending on the particle properties and the closure equations for the slip velocities considered, these systems of conservation laws are, in general, not hyperbolic. For $N = 2$ this means they can be of mixed hyperbolic-elliptic type. This is particularly likely to happen with suspensions whose particles differ in density. On the other hand, the analysis of [50] clearly shows that the two particular models (defined by the systems of slip velocities) considered in the cited contribution to this issue are both hyperbolic.

4.2.4. *Vessels with varying cross-section and centrifuges*

The basic assumption of Kynch's theory, namely that the local solid-fluid relative or drift velocity is a function of the solids concentration only, can also be applied to sedimentation processes in vessels with varying cross-section, and to centrifuges with a rotating frame of reference, if it is assumed that the gravitational body force can be neglected against the centrifugal force, and that Coriolis forces are unimportant. Both cases lead to equations similar to Equation (3) but that have additional smooth source terms. Solutions to these equations can still be determined by the method of characteristics, but the difficulty is that, in contrast to our previous discussion, characteristics and iso-concentration curves no longer coincide and the structure of global solutions is, in general, more complicated than in the standard case of batch settling in a cylindrical column. Anestis [51] and Anestis and Schneider [52] construct explicit weak (discontinuous) solutions in these cases. Their arguments determining whether a discontinuity is physically admissible arise from physical insight. Although the general existence and uniqueness result by Kružkov [53] admits a source term and therefore includes the models studied in [51, 52], it still remains to be shown that the constructed solutions are indeed entropy-weak solutions.

4.2.5. *Several space dimensions*

The preceding extensions of Kynch's sedimentation model all refer to one space dimension. A natural question is whether there exists a straightforward extension to several space dimensions. Unfortunately, the appropriate answer seems to be negative. This can be inferred from the fact that the model arises from the solid and fluid mass balances,

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{v}_{\mathrm{s}}) = 0, \quad \frac{\partial \phi}{\partial t} - \nabla \cdot \big((1-\phi)\mathbf{v}_{\mathrm{f}}\big) = 0, \tag{13}$$

where $\mathbf{v}_{\mathrm{s}}$ and $\mathbf{v}_{\mathrm{f}}$ are the solid and fluid phase velocities. Equations (13) are equivalent to

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{v}_{\mathrm{s}}) = 0, \quad \nabla \cdot \mathbf{q} = 0, \tag{14}$$

where $\mathbf{q} = \phi \mathbf{v}_{\mathrm{s}} + (1-\phi)\mathbf{v}_{\mathrm{s}}$ is the volume average velocity of the mixture. Only in one space dimension the flow field q and the concentration distribution ϕ can be determined from (14) if the slip velocity $\mathbf{v}_{\mathrm{r}} = \mathbf{v}_{\mathrm{s}} - \mathbf{v}_{\mathrm{f}} = \mathbf{v}_{\mathrm{r}}(\phi)$ and initial and boundary conditions are prescribed. In two or more space dimensions, additional equations for the motion of the mixture, *i.e.* for the velocity field $\mathbf{q}$, have to be solved. In order to obtain a well-posed system, this requires the inclusion of viscous effects. Suitable model equations were formulated by the authors in [49] and partly analyzed in [53].

4.3. APPLICATIONS

4.3.1. *Design of continuous thickeners*

The first paper following the publication of Kynch's [1] is that by Talmage and Fitch [54], which appeared in 1955. Using Kynch's theory and in conjunction with the cited treatments by Mishler [8] and Coe and Clevenger [12], they devise a method to derive the thickener area required to produce a sediment of given concentration at given solids handling rate. This method is described in detail in [10, 55]. Although Kynch's theory can not be regarded as an appropriate model for flocculent suspensions, thickener manufacturers still use and recommend Talmage and Fitch's method for design calculations [56].

4.3.2. *Other unit operations*

In the extension to continuous sedimentation the kinematic sedimentation model is used to describe the solid-fluid relative motion under the condition of a 'bulk' or 'plug' flow of the mixture, which can be oriented along or against the direction of gravity. Configurations of the latter case also occur in *fluidization*. In this operation, the fluid is pumped from below into a column with a settled bed of solids in order to resuspend the particles. Kynch's essential assumption, *i.e.* that the solid-fluid drift velocity is a function of the solids concentration only, was also independently stated by several authors in the 1950s [57–59], and is also one of the key ingredients of the recent treatment by Thelen and Ramirez [60]. However, a mathematical analysis of corresponding fluidization models is still lacking. This is possibly not an entirely straightforward-extension of the sedimentation analysis. For example, the stability proof (quoted in Bürger and Karlsen's paper of this issue) requires that q and f_{bk} have the same sign, which is *not* the case in fluidization. On the other hand, complex *in* stability phenomena do indeed occur in fluidization, as discussed in the recent book by Jackson [61].

At high solids concentrations, the Kynch batch flux density function determining the solid-fluid relative velocity can be interpreted as a formula predicting the local permeability of a sediment layer. In fact, from a generalized Darcy's law [62] it can be derived [63] that the permeability $K = K(\phi)$ and the flux density function f_{bk} are related by

$$f_{\mathrm{bk}}(\phi) = -\frac{K(\phi)\Delta\varrho g\phi^2}{\mu_{\mathrm{f}}},$$

where $\Delta\varrho$ is the solid-fluid density difference, g is the acceleration of gravity, and μ_{f} is the dynamic viscosity of the pure fluid. Thus Kynch's sedimentation model also handles flow through porous media formed by solid particles, which is the basic principle of filtration processes. In fact, still adding the terms accounting for compressibility effects, one obtains an integrated model for pressure filtration with simultaneous sedimentation [64, 65]. Mathematically, the application of a pressure, for example through a piston, which reduces the balanced mixture volume in a way that depends on the porosity of the filter cake and thereby on the solution itself, leads to a free-boundary-value problem [65].

We finally mention that the extension to polydisperse suspensions has also paved the way to operations in which the differential settling behaviour of particles with different sizes or densities is important. Most notably, Lee [66] and Austin *et al.* [67] formulate a mathematical model of classification of solid particles.

4.3.3. *Areas of application*

The previous discussion has considered various mathematical extensions of the sedimentation model and practical uses for unit operations, and has focused on sedimentation in mineral processing. However, papers explicitly referring to [1] and utilizing Kynch's model (or one of its extensions) also arise from many other areas. Most notably, it is widely used in theories of continuously operated secondary wastewater settling tanks and (if the discussion is limited to steady states) frequently referred to as solids flux theory [68–73]. Other applications are soil consolidation problems in geotechnical engineering [74, 75]. The extension to polydisperse suspensions has been considered in volcanology ([76], although this paper does not refer to Kynch) and as a model for the production of functionally graded materials by casting of polydisperse suspensions [77, 78]. The theory has also been applied to blood sedimentation, where the relevant settling velocity is the so-called *Erythrocyte Sedimentation Rate* (ESR) [79, 80]. An extension to bidisperse sedimentation was suggested to study the differential settling behaviour of red and white blood cells [81]. These references illustrate that, although its idealizing assumptions are seldom satisfied, Kynch's theory has turned out to be a useful approximation for the sedimentation of suspensions in diverse areas.

5. This issue

In this special issue we present nine papers that consider different aspects of mathematical models for sedimentation and suspension flows. The first three deal with extensions and refinements of Kynch's sedimentation model, and are related to spatially one-dimensional setups. In his paper *Operating charts for continuous sedimentation I: control of steady states* S. Diehl applies his previous analyses of the continuous sedimentation model with discontinuous flux function to construct systematically charts of steady-state solutions, which contain all information necessary to control continuous sedimentation under given control objective formulated in terms of the output variables in steady state. In doing so he exploits the main advantage of using Kynch's theory, which is the possibility to construct *exact* weak solutions to the resulting first-order conservation law.

In general, exact solutions can not be obtained within the framework of the second paper, *On some upwind difference schemes for the phenomenological sedimentation-consolidation model*, by R. Bürger and K.H. Karlsen, who study the discussed extension to flocculated suspensions. They briefly review the mathematical analysis of the resulting strongly degenerate convection-diffusion problem and present a numerical scheme which approximates the right physically relevant solution of the problem, taking into account the degeneracy and possible discontinuity of the diffusion coefficient.

The value of modern high-resolution schemes to solve the conservation equations occurring in the context of sedimentation models is also illustrated in the contribution *Central-difference solutions of the kinematic model of settling of polydisperse suspensions and three-dimensional particle-scale simulations* by R. Bürger, K.-K. Fjelde, K. Höfler and K. H. Karlsen, which deals with the kinematic models for polydisperse sedimentation that give rise to first-order systems of conservation laws.

Spatially one-dimensional sedimentation models are useful in such configurations where the flow of the mixture is essentially parallel to the acting body force. However, the consolidation rate of a highly concentrated flocculated suspension can be enhanced by the application of shear. K. Gustavsson and J. Oppelstrup in their contribution *Numerical 2D models of consol-*

idation of dense flocculated suspensions present numerical solutions of a two-dimensional mathematical model which includes appropriate equations for the motion of the mixture, as discussed in Section 4.2.5. In particular, different viscosity models for the mixture are considered.

The first four papers adopt the Theory of Mixtures and model both the solid particles and the fluid as continua, which is a useful approximation for the computation of macroscale behaviour. The remaining five contributions take into account (in different manners) the behaviour of individual, dispersed particles or drops. In his contribution *Numerical simulation of sedimentation in the presence of 2D compressible convection and reconstruction of the particle-radius distribution function* K. V. Parchevsky considers the sedimentation of a dilute polydisperse suspension under the effect of heat-driven convection, where the particle-size distribution is to be determined. In turns out that convection acts as a size filter separating particles on the basis of their radii.

In the preceding contributions the particles (or particle flocs) are, for simplicity, assumed to be spherical. For non-spherical particles, the orientation with respect to the body force has an appreciable effect on the settling velocity. In their paper *Computation of settling speed and orientation distribution in suspensions of prolate spheroids* E. Kuusela, K. Höfler and S. Schwarzer present an efficient numerical technique for the simulation of the sedimentation of such non-spherical particles.

So far all papers treat suspensions of solid particles in a viscous fluid. A different type of mixture are emulsions, in which the dispersed phase is an insoluble gas or liquid forming bubbles or drops. The rheological properties of emulsions largely depend on the deformations the drops or bubbles can undergo, and on the microstructures they form. These deformations are in turn determined by the distribution of the surfactant concentration on the surface of each bubble or drop. This effect is investigated numerically in the paper *Numerical investigation of the effect of surfactants on the stability and rheology of emulsions and foam* by C. Pozrikidis.

The mathematical models used in the first four contributions of this special issue are based on the Theory of Mixtures with the implicit assumption that the size of individual particles is negligibly small. Models of flows of suspensions with relatively large particles are analyzed in the last two contributions. In *A turbulent dispersion model for particles or bubbles* D. A. Drew derives a model for dispersed two-phase flow including the source of dispersion. In particular, turbulence is included. In their paper *Average pressure and velocity fields in non-uniform suspensions of spheres in Stokes flow*, M. Tanksley and A. Prosperetti address the problem of defining the right mixture pressure and velocity fields for non-uniform suspensions of rigid spheres in Stokes flow.

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